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Precession of Kepler's orbit

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The motion of a point-mass object of mass m into and out of a system S consisting of a large number of point particles uniformly distributed inside a sphere of radius R, under the force of gravity, is investigated in this article. We found that the path of the object, if allowed to penetrate into the system without collision, is no longer a stationary Kepler's orbit. The major axis of the orbit is found to be precessing about an axis passing through the center of the system and perpendicular to the plane of motion of the object. Under certain conditions, the object will return back to its initial location after a definite number of cycles of revolution. A star moving into and out of a galaxy would be a practical example of our analysis.

We start our analysis by using a polar coordinate system with its origin at the center of the system of mass M, $m \ll M$. Only the weak gravitational field is considered and the relativistic effect has been neglected. This means

$$GM/Rc^2 \ll 1, \tag{1}$$

where c is the speed of light in vacuum and G the gravitational constant. Using the law of universal gravitation, the force F acting on the object, called object P, and its potential energy V can be written as

$$\mathbf{F} = \begin{cases} -k\mathbf{r}/R^3, & r \leqslant R, \\ -k\mathbf{r}/r^3, & r > R, \end{cases} \tag{2}$$

$$V = \begin{cases} kr^2/2R^3 - 3k/2R, & r \leqslant R \\ -k/r, & r > R, \end{cases} \tag{3}$$

where $k \equiv GMm$ and r is the distance between the origin and the object. For the case of $r \geqslant R$, the equation of the orbit has the form[1]

$$r^{-1} = (mk/l^2)[1 + \epsilon\cos(\theta - \theta_0')], \tag{4}$$

where

$$\epsilon = (1 + 2El^2/mk^2)^{1/2}, \tag{5}$$

E is the total energy, l the total angular momentum, and θ the angle between the position vector $\mathbf{r}$ and the polar axis OO'.

θ_0' in Eq. (4) can be determined by the initial conditions. Initially, we let $\theta_0' = 0$, and (4) reduces to

$$r^{-1} = (mk/l^2)(1 + \epsilon\cos\theta). \tag{6}$$

For the case of $r < R$, we have[1]

$$\begin{aligned} \theta &= \theta_0 - \int (2mE/l^2 - 2mV/l^2 - u^2)^{-1/2}\, du \\ &= \theta_0 - \int \left[\frac{2mE}{l^2}\left(1 + \frac{3k}{2RE}\right) - \frac{mk}{l^2R^3u^2} - u^2\right]^{-1/2} du, \end{aligned} \tag{7}$$

where $u = r^{-1}$, and θ_0 is a constant of integration.

With the substitution of $x = u^2$, (7) simplifies to

$$\theta = \theta_0 - \tfrac{1}{2}\sin^{-1}[(2x - B)/(B^2 + 4A)^{1/2}], \tag{8}$$

where

$$A = -mk/l^2R^3, \tag{9}$$

$$B = (2mE/l^2)(1 + 3k/2RE), \tag{10}$$

and

$$r^{-2} = B/2 + \tfrac{1}{2}(B^2 + 4A)^{1/2}\sin[2(\theta_0 - \theta)]. \tag{11}$$

Equation (11) represents the orbit equation of the particle inside S. The θ_0 can be determined by the initial conditions.

As shown in Fig. 1, at the initial point of entry of P into S, $r = R$, $\theta = \theta_1$. We get, from Eqs. (6) and (11),

$$\theta_1 = \cos^{-1}[\epsilon^{-1}(l^2/mkR - 1)], \tag{12}$$

$$R^{-2} = B/2 + \tfrac{1}{2}(B^2 + 4A)^{1/2}\sin[2(\theta_0 - \theta_1)], \tag{13}$$

or

$$\theta_0 = \theta_1 + \tfrac{1}{2}\sin^{-1}[(2/R^2 - B)/(B^2 + 4A)^{1/2}]. \tag{14}$$

The minimum value of r, $r_{\min}$, can be determined by letting $\sin[2(\theta_0 - \theta)] = 1$ in Eq. (11). We have

$$r_{\min} = [2/(B + \sqrt{B^2 + 4A})]^{1/2} \tag{15}$$

and the value of θ at $r = r_{\min}$ is given by the equation

$$\theta(r_{\min}) = \theta_0 - \pi/4. \tag{16}$$

At the point of exit of P from S, we have $r = R$, $\theta = \theta_2$. Using Eq. (13), we must have

$$\sin[2(\theta_0 - \theta_1)] = \sin[2(\theta_0 - \theta_2)]. \tag{17}$$

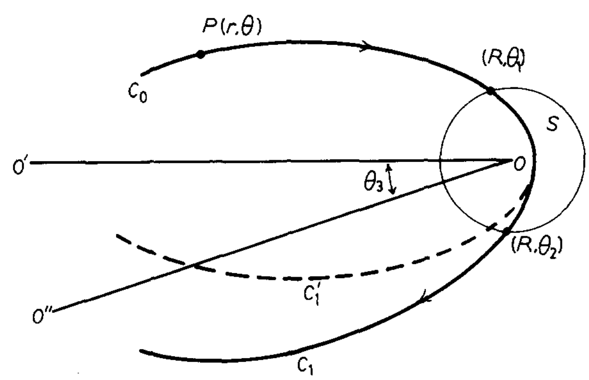

Fig. 1. Orbit of particle P around a system S consisting of a large number of point particles uniformly distributed within a sphere of radius R.

The nontrivial solution of (17) is

$$2(\theta_0-\theta_1)=\pi-2(\theta_0-\theta_2)$$

or (18)

$$\theta_2=2\theta_0-\theta_1-\pi/2.$$

Notice that if all the mass of S were concentrated at the origin, the orbit of P would be a Kepler's orbit and $\theta_1=-\theta_2$. From (18), we see that in general, $\theta_2\neq-\theta_1$.

Because the total energy and angular momentum of the particle P are conserved, the orbit of the particle after it leaves S will again be determined by Eq. (4) and θ'_0 will no longer be zero if $\theta_2\neq-\theta_1$. Replacing θ'_0 by θ_3, we can rewrite Eq. (4) as

$$r^{-1}=\frac{mk}{l^2}[1+\epsilon\cos(\theta-\theta_3)]. \quad (19)$$

At the point of exit of the particle from the system S, as shown in Figs. 1 and 2, we have $\theta=\theta_2$ and $r=R$. Equation (19) now becomes

$$R^{-1}=(mk/l^2)[1+\epsilon\cos(\theta_2-\theta_3)]$$

or

$$\theta_3=\theta_2-\cos^{-1}[\epsilon^{-1}(l^2/mkR-1)]=\theta_2\pm\theta_1. \quad (20)$$

From Fig. 1, we see that if $\theta_2=-\theta_1$, $\theta_3=0$. This means that we must use a plus sign in Eq. (20). We then have, using Eq. (18),

$$\theta_3=\theta_2+\theta_1=2\theta_0-\pi/2. \quad (21)$$

The θ_3 in general is different from zero. So the orbit of the particle does have a precession.

The Kepler's orbit C'_1 of Eq. (6) differs from our derived orbit of Eq. (19) as shown in Fig. 1. The angle between the major axis of C_1 and that of C'_1 is θ_3, a nonzero value. The θ_3 depends on the values of m, M, E, l, and R. As shown in Fig. 2, for $E<0$, if we use OO'' as a new polar axis and repeat the above calculations, we can get the orbit of the second cycle. The orbit of the nth cycle can then be expressed as

$$r_n^{-1}=(mk/l^2)[1+\epsilon\cos(\theta-n\theta_3)], \quad r>R, \quad (22)$$

$$=r_n^{-2}=B/2+\tfrac{1}{2}(B^2+4A)^{1/2}\times\sin\{2[\theta_0-(\theta-n\theta_3)]\}, \quad r\leqslant R], \quad (23)$$

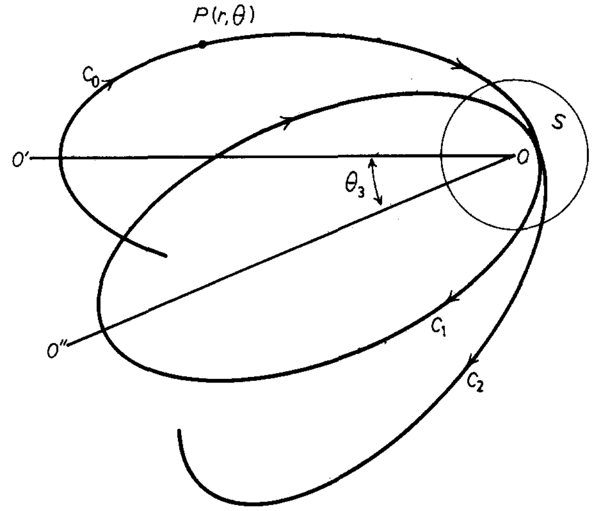

Fig. 2. Precession of the orbit of particle P for $E<0$.

where $n=0, 1, 2 \ldots$.

Equations (22) and (23) give us the precession of Kepler's orbit. The orbit of the particle will be closed after n cycles if condition $n\theta_3=2\pi$ is satisfied. Using Eqs. (12), (14), and (21), the condition $n\theta_3=2\pi$ can be expressed in the following form:

$$2\cos^{-1}\left[\frac{1}{\epsilon}\left(\frac{l^2}{mkR}-1\right)\right]+\sin^{-1}\left(\frac{2/R^2-B}{(B^2+4A)^{1/2}}\right)=\left(\frac{1}{n}+\frac{1}{4}\right)2\pi. \quad (24)$$

The orbit of the particle P, for the case of $E<0$, is shown in Fig. 2.

[1]H. Goldstein, *Classical Mechanics* (Addison–Wesley, Reading, MA, 1965), pp. 73–78.